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TWO-SIDED ESTIMATES ON THE ATTAINABILITY DOMAINS OF CONTROLLED SYSTEMS*

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The differential equations describing the evolution of outer and inner ellipsoidal approximations to the attainability domains of controlled systems are studied. Estimates of the volumes are obtained and certain extremal properties of these ellipsoids are formulated.

Knowledge of attainability domains is required not just in control problems. It is essential also when estimating the influence of perturbations on dynamic systems under a estimation (filtration) of the dynamic systems in the presence of measurement errors /1 – 3/, as well as in differential games /4/. However, the effective construction of attainability domains is very difficult. An explicit description is available only in the simplest cases, while a numerical construction /5,6/ entails the need for approximating the domain's boundary in a multidimensional space. In a number of cases /7/ the evolution of the attainability domain is described by an ordinary differential equation in an infinite-dimensional space. It is natural to try to find a reasonable finite-dimensional approximation of this complex dynamic system. Ordinary differential equations were obtained in /8 – 10/, describing the evolution of outer and inner ellipsoidal approximations to the attainability domains.

1. Evolution of attainability domains. Consider a controlled dynamic system of general form

$$x^{\cdot} = f(x, u, t), \quad u \in U(t), \; x(s) \in D(s) \tag{1.1}$$

Here t is time, $x \in \mathbf{R}^n$ is the phase vector, $U(t)$ are prescribed sets of values of control u, $D(s)$ is the domain in which the phase vector lies at the initial instant s (the initial domain). The set of endpoints $x(t)$ of the trajectories starting off in $D(s)$ and satisfying (1.1) for some admissible control $u(t)$ is called the attainability domain $D(t) = D(s, t)$ of system (1.1) at the instant t. In order to emphasize the dependence of the attainability domain on the initial set D and on the initial instant s we write $D(t) = D(t, s, D(s))$. The attainability domain is an important characteristic of a controlled system, permitting the evaluation of control capability, as well as essentially simplifying the solving of optimization problems. Thus, the problem of minimizing a terminal functional $F(x(T))$, where T is a fixed instant and $F(x)$ is a prescribed function, is equivalent to seeking the minimum of $F(x)$ on the attainability domain $D(T)$.

Let us consider a controlled system (1.1) in which

$$f(x, u, t) = A(t)x + g(x, u, t) \tag{1.2}$$

where $A(t)$ is an $n \times n$-matrix, while the set

$$G(x, t) = \{y \in \mathbf{R}^n;\ y = g(x, u, t),\ u \in U(t)\}$$

of values of $g(x, u, t)$ as a function of u admits of the two-sided estimate $G_-(t) \subset G(x, t) \subset G_+(t)$, , where $G_\pm(t)$ are closed convex sets. Let an analogous estimate $D_-(s) \subset D(s) \subset D_+(s)$ on the initial domain exist as well. Then, obviously, the attainability domains $D_\pm(t)$ of the linear controlled systems

$$x^{\cdot} = A(t)x + u_-, \quad x(s) \in D_-(s), \quad u_- \in G_-(t)$$
$$x^{\cdot} = A(t)x + u_+, \quad x(s) \in D_+(s), \quad u_+ \in G_+(t)$$

yield the two-sided estimate $D_-(t) \subset D(t) \subset D_+(t)$ for the required attainability domains of system (1.1). Consequently, the problem of estimating the attainability domains for system (1.1), (1.2) is to a significant extent reduced to the analogous problem for the linear system

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$$x^{\cdot} = A(t)x + u,\ x(s) \in D(s),\ u \in U(t) \tag{1.3}$$

where $D(s)$ and $U(t)$ are prescribed closed convex sets in $\mathbf{R}^n$. This problem is precisely the one to be studied below. We describe the evolution of the attainability domains $D(t)$ of system (1.3) with the aid of an ordinary differential equation in an infinite-dimensional space. For this we note that $D = D(t)$ is a closed convex set, and such sets are uniquely specified by their support function /11/

$$H(\xi) = H_D(\xi) = \sup_{x \in D}(x, \xi)$$

where $\xi \in \mathbf{R}^n$ and $(,)$ is the scalar product. The following theorem describes the evolution of the support functions of the attainability domains of system (1.3).

Theorem 1. Let $H(t, \xi) = H_{D(t)}(\xi)$ be the support function of the attainability domain $D(t)$ of system (1.3) and $h(t, \xi) = H_{U(t)}(\xi)$ be the support function of the control domain $U(t)$. Then

$$\frac{\partial H}{\partial t}(t, \xi) = \left(A(t)\frac{\partial H}{\partial \xi}(t, \xi), \xi\right) + h(t, \xi) \tag{1.4}$$

where $\partial/\partial\xi$ denotes the gradient with respect to ξ.

The proof of Theorem 1 is obtained by a differentiation of the support function, and is omitted here.

Notes. 1°. An equation of type (1.4) was obtained in /7/ under somewhat different assumptions.

2°. In general, the gradient $\partial H/\partial\xi$ does not exist at all points. However, the definition of the expression $(\partial H/\partial\xi(\xi), \eta)$, if it is understood as a directional derivative along η (i.e., $\lim \varepsilon^{-1}(H(\xi + \varepsilon\eta) - H(\xi))$ as $\varepsilon \downarrow 0$), is everywhere well-posed for any convex function $H(\xi)$ /11/. The term $(A(\partial H/\partial\xi), \xi) = (\partial H/\partial\xi, A^* \xi)$ in Eq.(1.4) should be understood in precisely this sense (matrix A^* is the transpose of A).

Definition. Let $\Omega(t)$ be a family of closed convex sets depending on a parameter $t \geqslant s$, $H(t, \xi) = H_{\Omega(t)}(\xi)$ be the corresponding family of support functions. We say that $\Omega(t)$ is family of subattainability domains (respectively, superattainability domains) for system (1.3) if the differential inequality

$$\frac{\partial H}{\partial t}(t, \xi) \leqslant \left(A(t)\frac{\partial H}{\partial \xi}(t, \xi), \xi\right) + h(t, \xi),\ H(s, \xi) = H_{D(s)}(\xi) \tag{1.5}$$

(respectively, the inequality (1.5) with sign $\geqslant$) is fulfilled.

We note that the inclusion $\Omega_1 \subset \Omega_2$ of closed convex sets corresponds precisely to the inequality $H_{\Omega_1}(\xi) \leqslant H_{\Omega_2}(\xi)$ for their support functions. Therefore, from Theorem 1 it follows that $\Omega(t) \subset D(t)$ in the case of subattainability domains and $\Omega(t) \supset D(t)$ in the case of superattainability domains. Further, if $\tau \leqslant t$ and $D(\tau, t)$ is the attainability domain at instant t of the controlled system

$$x^{\cdot} = A(t)x + u,\ x(\tau) \in \Omega(\tau),\ u \in U(t) \tag{1.6}$$

then $D(\tau, t)$ contains the subattainability domain $\Omega(t)$ (respectively, is contained in the superattainability domain $\Omega(t)$). As a matter of fact the condition of subattainability of family $\Omega(t)$ is equivalent to the conditions $\Omega(s) = D(s)$ and $D(t, \tau, \Omega(\tau)) \supset \Omega(t)$ for any $\tau < t$. In such a form the subattainability condition is suitable also for nonlinear controlled systems. The substitution of the inclusion in the second condition to inverse leads to the superattainability condition.

The problem on the approximation of the attainability domains of the form prescribed can now be stated exactly. The volume of set Ω is denoted $V(\Omega)$.

Definition. Let M be some class of convex sets. We say that the family $\Omega(t) \in M$ of subattainability (respectively, superattainability) domains locally best approximates the family of attainability domains $D(t)$ of system (1.3) if at any instant τ the derivative of the volume

$$d/dt\, V(\Omega(t))|_{t=\tau}$$

reaches the maximum (minimum) among all families of subattainability (superattainability) domains from M for the system (1.6).

We are required to construct effectively the locally best family of sub- and superattainability domains of class M. This problem is solved in Sect.2 for the case when M is the class of all ellipsoids.

2. Ellipsoidal approximation.

We introduce the notation $E(a, Q)$ for the ellipsoid (Q is a positive-definite symmetric matrix)

$$\{x \in \mathbf{R}^n, \quad (Q^{-1}(x-a), \ x-a) \leqslant 1\}, \quad a \in \mathbf{R}^n$$

We assume that at the initial instant $s = 0$ the initial domain $D(0)$ of controlled system (1.3) is the ellipsoid $E(a_0, Q_0)$ and that the control domain $U(t)$ is the ellipsoid $E(b(t), G(t))$. Then the evolution of the ellipsoids locally best approximating the attainability domains of the system

$$x^{\cdot} = A(t)x + u, \ x(0) \in E(a_0, \ Q_0), \ u \in E(b(t), \ G(t)) \tag{2.1}$$

is descirbed by differential equations presented in Theorem 2 (given here without proof).

Theorem 2. For the family of ellipsoids $E(a_{\mp}(t), Q_{\mp}(t))$ to be the locally best family of subattainability (superattainability) ellipsoids for system (2.1), it is necessary and sufficient that the vector $a_{\mp}(t)$ and the matrix $Q_{\mp}(t)$ be the solution of the following Cauchy problem:

in the subattainability case

$$\begin{aligned} &a_-^{\cdot} = A(t)a_- + b(t), \quad a_-(0) = a_0 \\ &Q_-^{\cdot} = A(t)Q_- + Q_-A(t)^* + 2R^{-1}(RQ_-R^*)^{1/2}(RG(t)R^*)^{1/2}R^{*-1} \\ &Q_-(0) = Q_0 \end{aligned} \tag{2.2}$$

where R is a nondegenerated matrix such that RQ_-R^* and $RG(t)R^*$ are diagonal matrices:

in the superattainability case

$$\begin{aligned} &a_+^{\cdot} = A(t)a_+ + b(t), \quad a_+(0) = a_0 \\ &Q_+^{\cdot} = A(t)Q_+ + Q_+A(t)^* + hQ_+ + h^{-1}G(t), \quad Q_+(0) = Q_0 \\ &h = [n^{-1}\operatorname{Tr}(Q_+^{-1}G(t))]^{1/2} \end{aligned} \tag{2.3}$$

Obviously, $a_-(t) \equiv a_+(t)$. Below we denote $a_-(t) = a_+(t) = a(t)$.

3°. It is well known (see /12/, for example) that if A and B are symmetric matrices one of which is positive definite, then a nondegenerated matrix R exists such that RAR^* and RBR^* are diagonal. In spite of the possible ambiguity in the choice of R the expression $R^{-1}(RAR^*)^{1/2} \times (RBR^*)^{1/2}R^{*-1}$ depends only on A and B and equals $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ if matrix A is positive definite. Therefore, the right-hand side of (2.2) is correctly defined.

4°. Equations (2.2) and (2.3) were obtained earlier in /8–10/ from other considerations. Theorem 2 establishes the extremal properties of the ellipsoids described by these equations.

3. Evolution of the volumes of the inner and outer ellipsoids.

From Theorem 2 we can obtain formulas for the derivative of the volumes of the ellipsoids locally best approximating from above and from below the attainability domain of system (2.1).

Lemma (/12/, Chapter XV). Let $A(t)$ be a family of invertible matrices depending smoothly on t. Then

$$d/dt \ln \det A(t) = \operatorname{Tr}(A(t)^{-1}A^{\cdot}(t)) \tag{3.1}$$

Corollary. Let $E(a(t), Q(t))$ be a family of ellipsoids and $V(t)$ be its volume. Then

$$\frac{d}{dt}V(t) = \frac{1}{2}V(t)\operatorname{Tr}(Q(t)^{-1}Q^{\cdot}(t)) \tag{3.2}$$

Indeed, $V(t) = \omega_n[\det Q(t)]^{1/2}$, where ω_n is the volume of the unit ball $E(0, I)$ (I is the unit matrix). Therefore, (3.2) follows from (3.1). The result of Sect.3 is as follows.

Corollary from Theorem 2. Let $E(a(t), Q_{\mp}(t))$ be a locally optimal family of ellipsoids of subattainability (index minus) or superattainability (index plus) for system (2.1) and let $V_{\mp}(t)$ be the volume of these ellipsoids. Then, respectively, (Q_-^0 and G^0 are diagonal matrices)

$$d/dt \ln V_-(t) = \operatorname{Tr} A(t) + \operatorname{Tr}[(Q_-^0)^{-1/2}(G^0)^{1/2}] \tag{3.3}$$

$$Q_-^0 = RQ_-R^*, \quad G^0 = RG(t)R^*$$

$$d/dt \ln V_+(t) = \operatorname{Tr} A(t) + [n \operatorname{Tr}(Q_+(t)^{-1}G(t))]^{1/2} \tag{3.4}$$

Proof. We multiply the right-hand side of Eq.(2.2) by Q_-^{-1} and we compute

$$\operatorname{Tr} Q_-^{-1}Q_-^{\cdot} = 2[\operatorname{Tr} A + \operatorname{Tr}(Q_-^{-1}R^{-1}(Q_-^0)^{1/2}(G^0)^{1/2}R^{*-1})] = 2\{\operatorname{Tr} A + \operatorname{Tr}[(Q_-^0)^{-1/2}(G^0)^{1/2}]\}$$

Hence, with due regard to formula (3.2), we obtain (3.3). Analogously, in the case of system (2.3) we have

$$\operatorname{Tr} Q_+^{-1}Q_+^{\cdot} = \operatorname{Tr} [2A + hI + h^{-1}Q_+^{-1}G] = 2\{\operatorname{Tr} + [n\operatorname{Tr}(Q_+^{-1}G)]^{1/2}\}$$

From this and from (3.2) we obtain (3.4).

4. Comparison of the volumes of the inner and outer ellipsoids.

The next theorem gives a comparative estimate of the volumes V_+ and V_- of the approximating ellipsoids.

Theorem 3. In the above-adopted notation we have (V_0 is the volume of domain $D(0)$)

$$V_-(t) \leqslant V_+(t) \tag{4.1}$$

$$\frac{V_+(t)}{V_0(t)} \exp\left(-\int_0^t \operatorname{Tr} A(\tau)\,d\tau\right) \leqslant \left[\frac{V_-(t)}{V_0(t)} \exp\left(-\int_0^t \operatorname{Tr} A(\tau)\,d\tau\right)\right]^{\sqrt{n}}$$

Proof. The first inequality in (4.1) is trivial since the inner ellipsoid is contained in the outer. To prove the second we rewrite (3.3) and (3.4) as

$$\frac{d}{dt} \ln\left[V_-(t) \exp\left(-\int_0^t \operatorname{Tr} A(\tau)\,d\tau\right)\right] = \operatorname{Tr}\left(Q_-^0(t)^{-1/2} G^0(t)^{1/2}\right) \tag{4.2}$$

$$\frac{d}{dt} \ln\left[V_+(t) \exp\left(-\int_0^t \operatorname{Tr} A(\tau)\,d\tau\right)\right] = [n \operatorname{Tr}(Q_+(t)^{-1} G(t))]^{1/2} \tag{4.3}$$

and establish the inequality

$$\operatorname{Tr}[(Q_-^0)^{-1/2}(G^0)^{1/2}] \geqslant \operatorname{Tr}(Q_+^{-1}G) \tag{4.4}$$

connecting the right-hand sides of (4.2) and (4.3). To prove relation (4.4) we note that Q_-^0 and G^0, and, consequently, also $(Q_-^0)^{-1/2}, (G^0)^{1/2}$ are diagonal matrices with positive diagonal elements. If $\alpha = (\alpha_{ij})$ is any such matrix, then

$$(\operatorname{Tr}\alpha)^2 = \left(\sum_{i=1}^n \alpha_{ii}\right)^2 \geqslant \sum_{i=1}^n \alpha_{ii}^2 = \operatorname{Tr}(\alpha^2)$$

Therefore,

$$\{\operatorname{Tr}[(Q_-^0)^{-1/2}(G^0)^{1/2}]\}^2 \geqslant \operatorname{Tr}\{[(Q_-^0)^{-1/2}(G^0)^{1/2}]^2\} = \operatorname{Tr}[(Q_-^0)^{-1}G^0]$$

since Q_-^0 and G^0 are diagonal. From formulas (3.3) for these matrices follows $(Q_-^0)^{-1}G^0 = R^{*-1}Q_-GR^*$, and, therefore, $\operatorname{Tr}(Q_-^0)^{-1}G^0 = \operatorname{Tr} Q_-^{-1}G$. Thus, $(\operatorname{Tr}(Q_-^0)^{-1/2}(G^0)^{1/2})^2 \geqslant \operatorname{Tr}Q_-^{-1}G$, and to prove (4.4) it is enough to establish that

$$\operatorname{Tr} Q_-^{-1}G \geqslant \operatorname{Tr} Q_+^{-1}G \tag{4.5}$$

Since the inner ellipsoid is contained in the outer and shares a common center a then $\beta = Q_-^{-1} - Q_+^{-1}$ is a nonnegative-definite symmetric matrix: $(\beta x, x) \geqslant 0$, $Ax \in \mathbf{R}^n$. Matrix G is positive definite. Therefore, inequality (4.5) follows from the fact that

$$\operatorname{Tr}\beta G = \operatorname{Tr}\gamma\gamma^* = \sum_{i,j=1}^n \gamma_{ij}^2 \geqslant 0, \quad \gamma = \beta^{1/2}G^{1/2}$$

From (4.2)—(4.4) we obtain

$$\frac{d}{dt} \ln\left[V_+(t) \exp\left(-\int_0^t \operatorname{Tr} A(\tau)\,d\tau\right)\right] \leqslant n^{1/2} \frac{d}{dt} \ln\left[V_-(t) \exp\left(-\int_0^t \operatorname{Tr} A(\tau)\,d\tau\right)\right]$$

whence follows the second inequality in (4.1).

The proved Theorem 3 enables us to estimate from both sides one of the volumes V_+ or V_- of the ellipsoids if we know the other (V_- or V_+). Therefore, to obtain a two-sided estimate on the volume of the attainability domain $D(t)$ it is sufficient to integrate only one of the systems (2.2) or (2.3). It is interesting to compare inequality (4.1) with the following result Φ of John.

Theorem /13/. Let Ω be a centro-symmetric convex body centered at the origin. Then an ellipsoid $E \subset \Omega$ exists such that the ellipsoid $E^* = \sqrt{n}\,E = \{x \in \mathbf{R}^n,\ x = \sqrt{n}y,\ y \in E\}$ contains Ω.

5. Lower estimate on the inner ellipsoid's volume.

Using formula (4.2) we obtain a lower estimate on the volume of the locally optimal subattainability of ellipsoid.

Theorem 4. In the above-adopted notation the inequality

$$V_-(t)^{1/n} \geqslant \exp\left(\frac{1}{n}\int_0^t \operatorname{Tr} A(\tau)\,d\tau\right)\left[V_0^{1/n} + \int_0^t V_G(\tau)^{1/n} \exp\left(-\frac{1}{n}\int_0^\tau \operatorname{Tr} A(\sigma)\,d\sigma\right)d\tau\right] \tag{5.1}$$

is fulfilled, where $V_G(\tau)$ is the volume of the ellipsoid $E(b(\tau), G(\tau))$ of admissible controls (see (2.1)).

Proof. Let us estimate from below the right-hand side of formula (4.2); to be precise, let us show that

$$\operatorname{Tr}(Q_-^0)^{1/2}(G^0)^{1/2} \geqslant nV_G^{1/n}V_-^{-1/n} \tag{5.2}$$

Indeed, if α is a nonnegative-definite symmetric matrix and $\lambda_1, \ldots, \lambda_n$ are its eigenvalues, then

$$(1/n)\operatorname{Tr}\alpha = (1/n)\sum\lambda_i \geqslant (\lambda_1 \ldots \lambda_n)^{1/n} = (\det\alpha)^{1/n}$$

From this inequality for $\alpha = (Q_-^0)^{-1/2}(G^0)^{1/2}$ and from the definition of Q_-^0 and G^0 in (3.3) follows

$$\operatorname{Tr}(Q_-^0)^{-1/2}(G^0)^{1/2} \geqslant n\{\det[(Q_-^0)^{-1}G^0]\}^{1/(2n)} = n[\det(Q_-^{-1}G)]^{1/(2n)}$$

Hence follows (5.2). Substituting (5.2) into (4.2) and integrating from 0 to t, we obtain the assertion of Theorem 4.

Corollary. When $A \equiv 0$ we have the inequality

$$V(t)^{1/n} \geqslant \int_0^t V_G(\tau)^{1/n}\,d\tau$$

for the volume $V(t)$ of the attainability domain $D(t)$.

To prove this it is enough to apply inequality (5.1) and to note that $V(t) \geqslant V_-(t)$. This corollary can be obtained independently from the Brunn–Minkowski inequality

$$V(\Omega_1 + \Omega_2)^{1/n} \geqslant V(\Omega_1)^{1/n} + V(\Omega_2)^{1/n}$$

where $\Omega_1 + \Omega_2 = \{x \in \mathbf{R}^n,\ x = x_1 + x_2,\ x_i \in \Omega_i\}$ is the Minkowski sum of convex sets Ω_1 and Ω_2 /14/.

6. Upper estimate on the outer ellipsoid's volume.

We now obtain an upper estimate for the superattainability ellipsoid's volume. In contrast to the estimate in Theorem 4, in the estimate in Theorem 5 we now have, besides the matrices $A(t)$ and $G(t)$, also the derivative of the latter.

Theorem 5. Let $r(t) = G(t)^{1/2}$. Then in the adopted notation the inequality

$$V_+(t)^{1/n}\exp\left(-\frac{1}{n}\int_0^t \operatorname{Tr}A(\tau)\,d\tau\right) \leqslant V_0^{1/n}\left[1 + \left(\frac{\operatorname{Tr}Q_0^{-1}G(0)}{n}\right)^{1/2}\int_0^t \exp\left(\int_0^\tau c(\sigma)\,d\sigma\right)d\tau\right] \tag{6.1}$$

$$c(t) = \|r(t)^{-1}r^{\cdot}(t) - r(t)^{-1}A(t)\,r(t)\|$$

is fulfilled, where $\|\alpha\| = [\operatorname{Tr}(\alpha\alpha^*)]^{1/2}$ is the Hilbert–Schmidt norm of matrix α.

Proof. We esimate the right-hand side of formula (4.3). We set $P = rQ_+^{-1}r$. Then $\operatorname{Tr}P = \operatorname{Tr}Q_+^{-1}r^2 = \operatorname{Tr}Q_+^{-1}G$ and it all reduces to estimating $\operatorname{Tr}P$. Substituting $Q_+ = rP^{-1}r$ into differential Eq.(2.3), for P we obtain the equation

$$\begin{aligned} -P^{-1}P^{\cdot}P^{-1} + \{r^{-1}r^{\cdot}, P^{-1}\} = \{r^{-1}Ar, P^{-1}\} + n^{-1/2}(\operatorname{Tr}P)^{1/2}P^{-1} + \\ n^{1/2}(\operatorname{Tr}P)^{-1/2}I \quad (\{\alpha,\beta\} = \alpha\beta + \beta^*\alpha^*) \end{aligned} \tag{6.2}$$

After multiplication by P from the right and left, (6.2) becomes

$$P^{\cdot} = \{P, r^{-1}r^{\cdot} - r^{-1}Ar\} - n^{-1/2}(\operatorname{Tr}P)^{1/2}P - n^{1/2}(\operatorname{Tr}P)^{-1/2}P^2 \tag{6.3}$$

We set $B = r^{-1}r^{\cdot} - r^{-1}Ar$. Then from (6.3) we have

$$\frac{d}{dt}\operatorname{Tr}P = \operatorname{Tr}\{P, B\} - n^{-1/2}(\operatorname{Tr}P)^{3/2} - n^{1/2}(\operatorname{Tr}P)^{-1/2}(\operatorname{Tr}P^2) \tag{6.4}$$

For $\operatorname{Tr}\{P, B\} = 2\operatorname{Tr}PB$ we have the estimate

$$2\operatorname{Tr}PB \leqslant 2\|P\|\|B\| = 2c\|P\|$$

on the strength of the Cauchy–Buniakovskii inequality. In addition, $\|P\| = (\operatorname{Tr}P^2)^{1/2} = (\Sigma\lambda_i^2)^{1/2} \leqslant \Sigma\lambda_i = \operatorname{Tr}P$, where $\lambda_i \geqslant 0$ are the eigenvalues of the positive-definite matrix P. On the other hand, we have the inequality

$$\operatorname{Tr}P^2 = \Sigma\lambda_i^2 \geqslant (1/n)(\Sigma\lambda_i)^2 = (1/n)(\operatorname{Tr}P)^2$$

Taking account of the inequalities obtained and denoting $\operatorname{Tr}P$ by $\xi = \eta^{-2}$, we obtain

$$\eta^{\cdot} \geqslant -c(t)\,\eta + n^{-1/2} \tag{6.5}$$

Having made the change

$$\eta(t) = \zeta(t)\,\mu(t)^{-1}, \quad \mu(t) = \exp\int_0^t c(\tau)\,d\tau$$

in (6.5), we find $\zeta^{\cdot}(t) \geqslant n^{-1/2}\mu(t)$. Hence, by integrating, we obtain an estimate for $\zeta(t)$, and next for $\eta(t)$ and

$$[\mathrm{Tr}\,(Q_+^{-1}G)]^{1/2} = (\mathrm{Tr}\,P)^{1/2} \leqslant n^{1/2}\frac{d}{dt}\ln\left\{[\xi(0)]^{-1/2}n^{1/2} + \int_0^t \mu(\tau)\,d\tau\right\}$$

Now from formula (4.3) we find

$$\frac{d}{dt}\ln\left[V_+(t)\exp\left(-\int_0^t \mathrm{Tr}\,A(\tau)\,d\tau\right)\right]^{1/n} \leqslant \frac{d}{dt}\ln\left\{n^{1/2}[\xi(0)]^{-1/2} + \int_0^t \mu(\tau)\,d\tau\right\}$$

Integrating the last inequality with respect to t and taking the notation introduced into account, we obtain inequality (6.1). Theorem 5 is proved.

Theorems 3 – 5 yield two-sided estimates on the volumes of the outer and inner ellipsoids approximating the attainability domain, in terms of the parameters of the original controlled system.

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